

Identity and Ontology*

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Summary

A conception of numerical identity is introduced which, in accordance with a transcendental or imposition view of language, treats an identity predicate as having an ontologically generative function by genuinely being involved in the generation or construction of its domain of discourse. The proposed conception also allows for a plurality of identity predicates, each of which generating a domain, and it allows for the possibility that some such domains may not be unifiable with each other. All of these informal notions are explicated in formal terms. Finally, a comparison to Scott and Fourman's intuitionistic conception of identity is made.

1. Introduction

The work presented here has its origin in a project aiming at developing a formal representation of Henri Lauener's transcendental theory of language.² This theory, which in many respects is close to Ajdukiewicz's *Radical Conventionalism*,³ was conceived within a methodological framework called 'open transcendentalism'.⁴ As such, it embraces what is sometimes called an *imposition view* of language, i.e., roughly speaking, the view that language contributes in an essential way to the structure of reality. The two key tenets of this theory are (i) that semantic concepts, such as 'truth', 'falsity', 'meaning', 'analyticity' etc., must be relativised to what I call *language contexts*,⁵ and (ii) that

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² Lauener's views on the matter have been published in Lauener 1982-1992.

³ See, in particular, Ajdukiewicz 1934.

⁴ See, in particular, Lauener 1984, 1987, and 1990.

⁵ A language context, or "context" in Lauener terminology, is given by delineating some frame of activity (*Handlungszusammenhang*) such as the development of a scientific theory. The determination of such a frame, as seen by Lauener, involves, amongst other, the choice of a language system (including, in particular, the choice of a predicate scheme, and of domains of discourse) and the description of the frame's purpose. It is, of course, unlikely that all these features of Lauenerian contexts, particularly the pragmatic ones, can be usefully represented in a formal system. This is why we have decided to focus on the constituents he calls language systems.

the meaning of expressions is determined exclusively by their rule-governed use in the particular language context in question. The focus of this paper, however, will not be on these two principles,⁶ but on the semantic and ontological ‘underpinnings’ of Lauener’s view that a theory of language which is to avoid Quine’s well-known conclusions concerning reference, translation and analyticity must abandon the idea of a single, all-embracing universe of discourse and rather allow for domains of discourse which can be essentially disjoint, in the sense that they cannot be unified to form a joint domain. My aim here will thus be to suggest a conception of domains of discourse (and to develop an adequate formal, indeed model-theoretical representation thereof) which is compatible with Lauener’s views, and accordingly allows for the possibility of such essential disjointness.⁷

Classically, domains of discourse (or, for short, domains) are represented as sets or classes of featureless and unstructured objects (‘ur-elements’). Any two or more of these classes can always be set-theoretically unified to form a joint-class, and, as such, are classically taken to form the representation of a joint-domain.⁸ The classical conception of a domain underlying this model-theoretic representation is therefore clearly incompatible with the idea of essential disjointness. Does this mean that essentially disjoint domains cannot be represented in terms of set-theoretical constructs? To answer this, we must look more closely at the way in which domains are represented in classical model-theoretic semantics, and, indeed, at the way in which this representation is used for semantic purposes. Classically, I said, domains are represented as sets of featureless objects. This, however, is not completely accurate, for the elements of these sets are actually taken to be what I call *individuals*, by which I mean that they are assumed to be subject to a relation of numerical identity, usually expressed by “=”. In classical model theoretic semantics, the representation of a domain is used in two ways: on the one hand it provides the range of object-language referents, i.e. the range of objects which the individual constants and variables of the object-language can denote or refer to, on the other, it serves, as structured totality, to represent the semantics of the (ob-

⁶ For a formalisation of (i) and (ii) see Müller 1995.

⁷ To be quite clear, I do not intend to address the question whether or not Quine’s conclusions can be actually avoided by accepting this conception. This will have to be left for future investigation.

⁸ It is, of course, possible to introduce many-sorted semantics with distinct classical domains (See Barwise 1977, p. 42) in which each individual variable is restricted to ranging over some particular domain. But the very fact that these domains can always be set-theoretically unified to form joint-domains makes these restrictions purely cosmetic in nature: e.g. there is nothing *in the way* in which individuals are classically assigned to variables which might prohibit such assignments under certain circumstances (see also section 4.3).

ject-language) predicate of numerical identity. In the classical framework we thus find an intimate link between domains of discourse and numerical identity, and it is this very link which ensures that the classical conception of there being a single, universally applicable predicate of numerical identity goes hand in hand with the universal unifiability of domains. Indeed, in the classical framework, the semantics of numerical identity is based on the more fundamental conception of domains, i.e., to coin a phrase, “classically, identity is parasitic on ontology.”

This, in turn, suggests that it might be possible to achieve a model-theoretic representation of essentially disjoint domains if only one admits a conception of numerical identity which (i) allows for a plurality of identity predicates and for a possible essential incompatibility between them, and (ii) interprets the link between them and domains of discourse as ‘(ontologically) generative’ rather than ‘parasitic’. In other words, given the intrinsic connection between the classical representation of domains and numerical identity within the model-theoretic framework, my suggestion is that a representation of essentially disjoint domains in this framework is to be approached (i) by generalising the classical semantics of numerical identity predicates, and (ii) by considering the possibility that these predicates might be involved in the very characterisations (or, indeed, construction) of the domains they are associated with. The generalisation I have in mind is thus informally based on the following two ideas:

- (1) the classical formation of elementary identity sentences (i.e. sentences of the form “ $a = b$ ”) in terms of a *single* identity predicate expression (“=”) alone can lead to ambiguities,
and
- (2) elementary identity sentences can fail to have a truth-value even if these ambiguities are removed.

Take the following classically expressed elementary identity sentences:

- S_1 “Eiffel Tower = Tower of London”
 S_2 “9 = 7”

The identity symbol “=” in these sentences is, in my view, used ambiguously, in the sense that it must be associated with different predicates in its two occurrences. Why? For the simple reason that there are criteria (*‘identifying rules’*) which can be used, say, to ascertain that S_1 expresses a false statement – namely criteria involving the spatio-temporal location of the individuals referred to in S_1 – which cannot be applied to the individuals referred to in the

second sentence, because the individuals referred to S_2 do not have a spatial location. Thus 9 neither occupies the same spatial location as 7.⁹ The idea here is that if, for two elementary sentences " $P(a_1, a_2, \dots, a_n)$ " and " $P(b_1, b_2, \dots, b_n)$ " (where the referents of all the individual constants are well-established), there are criteria by means of which the truth or falsity of one can be established, but which are inapplicable in the case of the other, then " P " must mean something else in the first sentence than it does in the second.¹⁰

To avoid the ambiguity in S_1 and S_2 , let us replace " $=$ " with two distinct predicate expressions, " $=_1$ " and " $=_2$ " in the idea being, of course, that " $=_1$ " and " $=_2$ " have the same meaning as " $=$ " in S_1 and S_2 , respectively – and re-write the two sentences as

S_1^* "Eiffel Tower $=_1$ Tower of London"
 S_2^* "9 $=_2$ 7"

Assuming, for the moment that

(A) S_1^* and S_2^* are indeed elementary *identity* sentences, i.e. that " $=_1$ " and " $=_2$ " are predicate of numerical identity

– an assumption which, although plausible (given the genesis of S_1^* and S_2^*) I shall return to shortly – my claim concerning the possibility of truth-valueless elementary identity sentences follows for precisely the same reasons. Whereas S_1^* and S_2^* are truth-evaluable¹¹ the following sentences are not:¹²

S_3^* "Eiffel Tower $=_2$ Tower of London"
 S_4^* "9 $=_1$ 7"
 S_5^* "Eiffel Tower $=_2$ 7"
 S_6^* "Eiffel Tower $=_1$ 7"
 S_7^* "9 $=_1$ 9"

⁹ And the same can be said of the towers if we were to compare them numerically in terms of, say, prime factors.

¹⁰ Note that this includes the possibility that P is actually meaningless in the case where the criteria are inapplicable. It must be emphasised, however, that by assuming this sort of connection between the 'meaning' of (predicate) expressions and the 'verification criteria' of (elementary) sentences we do *not* commit ourselves to a verificationist view of meaning, in particular if, as is usual, verificationism is conceived of in terms of *empirical* verifications alone. The present argument simply assumes that there is a certain connection between the two (and it would be odd in the extreme, I believe, if there were no such connections at all) which allows us to enforce ambiguities, i.e. to conclude that an expression has been used ambiguously.

¹¹ Indeed they both express a *false* statement.

¹² Note that, given (A), all of these sentences must equally be elementary identity sentences.

Why? Each of them fails to be truth evaluable because the criteria associated with the identity predicate expression employed fail to be applicable to the referent of at least one of the occurring proper names.

Provided this sort of 'criteria-linked' semantic argument is acceptable, we can conclude that a plurality of identity predicates will always go hand in hand with there being truth-valueless elementary identity sentences. And this, in turn, suggests a way in which the essential disjointness of domains might be reflected in the semantics of numerical identity. Let us say that a collection C of objects is an $=_n$ -class (where $=_n$ is an identity predicate of some chosen language-system) if and only if all objects of C are significantly comparable by means of $=_n$ (meaning that " $x =_n y$ " is truth-evaluable for any assignment of objects from C to " x " and " y "). The idea then is, roughly, that the essential disjointness of an $=_n$ -class A and an $=_m$ -class B is semantically reflected in the fact that there is some a in A and some b in B for which " $a =_n b$ " (or " $a =_m b$ ") is without a truth-value. To be acceptable, this semantic characterisation of essential disjointness will require some further assumptions concerning the 'distribution' of truth-value gaps amongst elementary identity sentences (see 3.3.), but for the moment let us briefly consider its effects on classes associated with $=_1$ and $=_2$. Assuming still that these two predicates are identity predicates – relative to the (natural) language-system which we implicitly adopted to provide the meaning of S_1 and S_2 – the effect is simply that (within this language-system) we must distinguish at least two 'categories' of objects, namely those whose identity can be significantly compared to that of the Eiffel Tower, and those whose identity can be significantly compared to the number 9.¹³

Let us turn to the one hypothesis which has thus far been assumed throughout, namely that $S_1^* \dots S_7^*$ are *identity* sentences,¹⁴ i.e. that $=_1$ and $=_2$ are identity predicates. Given the orthodox view, enshrined in the classical axiom system for numerical identity and its model-theoretic implementation in the so-called standard semantics (for languages with identity), one might indeed still reasonably wonder what it could possibly mean to speak of *different* identity predicates (or, for that matter, of different identity relations)? And, of course, this will make no sense at all, if the notion of an "identity

¹³ To avoid misunderstandings, let me point out that it would be wrong to think that because the Eiffel Tower and the Tower of London are both towers we must interpret $=_1$ as an identity predicate which is somehow applicable only to towers. If anything we could paraphrase " $=_1$ " as "is material-object-identical with", but only insofar as "material object" is used to characterise the sort of objects which display the features required for an application of the identity-criteria associated with $=_1$.

¹⁴ The reason for dealing with these issues in terms of sentences and not statements is, of course, that it makes no sense to speak of statements without truth-values.

predicate" (or "identity relation") is used as a proper name. Yet, even though such a use might be defensible in the classical framework, it is possible to use it as a general term, designating all those predicates (relations) which satisfy what I call the theory of identity (see section 3.). To put it differently, the fact that under the assumptions implicit in the classical semantic framework, there is only one predicate (relation) satisfying this theory reflects only on these assumptions and does not mean that it is conceptually impossible to have more than one of these predicates (relations), as implied by the classical use of the term as proper name.

Indeed, another objection which might be raised to the earlier informal examples can be rejected in a similar fashion. What I have in mind is the view that sentences such as " $9 =_1 9$ ", if indeed truth-value less, cannot be an elementary *identity* sentence, because if $=_1$ is a *bona fide* identity predicate in the above mentioned sense, then by the law of reflexivity of identity, all sentences of the form " $x =_1 x$ " must be true. My reply to this is that the law of reflexivity of identity, as formulated in the classical theory of identity, is really a version of a more general law, a version which is tailor-made to suit the specific semantic assumptions implicit in the classical framework (such as the fact that it does not allow for truth-value gaps). This more general law does admit truth-value gaps even for sentences of the form " $x =_n x$ ".

Yet this reply will clearly only be satisfactory if we can provide a semantic framework and a theory of identity which allows for the sort of non-classical identity sentences informally introduced earlier on. The aim of sections 2 and 3 will be to achieve this in formal terms. In the hope that the formal account given in these two sections will be acceptable as an account of a generalised semantics for identity predicates, I shall then in section 4 turn to a formal representation of the initially mentioned generative character of identity predicates, and show that, when taken as ontologically generative, identity predicates do indeed give rise to *bona fide* essential disjointness. In section 5, finally, I shall try to argue for the usefulness of the conception of identity developed here by comparing it with the one put forward by Scott and Fourman in the context of intuitionistic logic.

2. The Atomistic¹⁵ Generalised Framework

Given that the classical (formal semantic) framework¹⁶ for, say, first-order¹⁷ predicate languages is a model-theoric one, it seemed sensible to try retain this model-theoric character for the envisaged generalised framework and meaning that it is to be given by a characterisation of models (section 2.2.) c and the specification of semantic valuation rules for syntactically well-formed formulae interpreted in these models (section 2.3.). It would not be difficult to specify these valuation rules for arbitrary first-order languages, but since for our purposes, we really need to consider only what might be called ‘purely denotational fragments’ of such languages, I decided, for the sake of notational simplicity, to restrict the following discussion to languages with the syntactic structure of this sort of first-order fragments.

2.1. The Syntactic Formation Rules

The syntax of such a purely denotational language (fragment) is given by a set $B = P \cup Var$ of *basic expressions* (the “vocabulary”) – where

$P = \bigcup_{i=1 \dots m} P_i$ is referred to as the “predicate scheme”,

$P_i = \{P_i, P'_i, P''_i, \dots, P_i^{(n_i)}\}$ (for $i = 1, \dots, m$) is the set of i -adic predicate constants; and

$Var = \{x, x', x'', \dots\}$ the set of individual variables

– and a single *syntactic formation rule*, namely that

if $P \in P_i$ then $P(x^{(k_1)}, \dots, x^{(k_i)})$ is a wff, for any $x^{(k_1)}, \dots, x^{(k_i)} \in Var$,

which defines the set **Wff** of (syntactically) well-formed formulae (“wffs”).

2.2. The Atomistic Generalised Conception of Formal Models

The conception of a formal model in the envisaged generalised framework is based on an arbitrarily given set \mathbf{M} from the realm of set-theory (whose ele-

¹⁵ The term “atomistic” is used here to indicate that the characterisation of the relevant models will be given (see section 2.2) exclusively in terms of *unstructured* individuals, i.e. that it will not refer to any “sub-ontological” features (see section 4.1).

¹⁶ My use of “framework” here is to introduce a distinction to semantics for particular languages. Such a (formal semantic) framework is – as we shall presently see – given by specifying, on the one hand, the nature of the formal models to be employed, and, on the other, certain rules concerning (i) the syntax of the languages to be considered, and (ii) the semantic evaluation of expressions of these languages.

¹⁷ The restriction to *first-order* languages is not crucial to the discussion to follow.

ments are referred to as *individuals*) which is used to define the class $\Sigma_{\mathbf{M}}$ of all ‘formal \mathbf{M} -interpretations’ σ , by which I mean functions of the following sort:

$$\sigma: \begin{cases} \text{Var} \rightarrow \mathbf{M}, \\ P_k \rightarrow 2^{M^k} \times 2^{M^k}; \text{ for } k = 1, \dots, m \end{cases}$$

Any sub-set \mathcal{M} of $\Sigma_{\mathbf{M}}$ is a *model* for \mathbf{B} (based on \mathbf{M}) if it is maximal with respect to the condition that

$$\sigma(P) = \sigma'(P) \text{ for all } \sigma, \sigma' \in \mathcal{M} \text{ and all } P \in \mathbf{P}.$$

The interpretations of a predicate *constant*, say P , within a model \mathcal{M} are thus all identical, which means that the denotation of P in \mathcal{M} can be represented as $\langle \mathbf{P}^{\mathcal{M}}, \mathbf{non-P}^{\mathcal{M}} \rangle$, where, for reasons which will become clear shortly, $\mathbf{P}^{\mathcal{M}}$ might be called the *positive component* (or “extension”) and $\mathbf{non-P}^{\mathcal{M}}$ the *negative component* of the denotation of P in \mathcal{M} , while $\mathbf{P}^{\mathcal{M}} \cup \mathbf{non-P}^{\mathcal{M}}$ could be referred to as the *significance-range* of P in \mathcal{M} . Let me refer to the collection of *all* these models (for \mathbf{B} , based on \mathbf{M}) as \mathbf{M}_B (or \mathbf{M} , if a particular vocabulary is fixed in advance).

2.3. Semantic Valuation Rules

In the classical framework, the ‘semantic valuation rules’ are simply the rules which specify for each wff a *unique* truth value for any given model. Since the envisaged generalised framework is to admit truth-value gaps, it is self-evident that the particular character given to these rules in the classical framework must also be generalised. This can be achieved if one assumes that semantic valuation rules, in general, define a *valuation relation* v between ‘semantically interpreted wffs’ $\langle \Phi, \mathcal{M}, \sigma \rangle$ (with $\Phi \in \mathbf{Wff}$ and $\sigma \in \mathcal{M} \in \mathbf{M}$) and ‘semantic values’ $\mathbf{S-Val} = \{T, F, N\}$ – where T is meant to stand for *true*, F for *false*, and N for *neither true nor false* (or ‘non-significant’).¹⁸

Given the simplicity of the purely denotational languages we are considering here, all we need is a single one of these rules, which (for, say $P \in \mathbf{P}$, $x \in \text{Var}$, and $\sigma \in \mathcal{M} \in \mathbf{M}$) can be formulated as:

$$\begin{aligned} v(\langle P(x), \mathcal{M}, \sigma \rangle, T) & \text{ iff}_{df} \sigma(x) \in \mathbf{P}^{\mathcal{M}} \\ v(\langle P(x), \mathcal{M}, \sigma \rangle, F) & \text{ iff}_{df} \sigma(x) \in \mathbf{non-P}^{\mathcal{M}} \\ v(\langle P(x), \mathcal{M}, \sigma \rangle, N) & \text{ iff}_{df} \sigma(x) \notin \mathbf{P}^{\mathcal{M}} \cup \mathbf{non-P}^{\mathcal{M}} \end{aligned}$$

– where $v(\langle P(x), \mathcal{M}, \sigma \rangle, T)$, $v(\langle P(x), \mathcal{M}, \sigma \rangle, F)$, and $v(\langle P(x), \mathcal{M}, \sigma \rangle, N)$ are meant to reflect the fact that “ $P(x)$ ” (under the interpretation $\sigma \in \mathcal{M}$) is *true*,

¹⁸ Given the formal character of these models, it would perhaps be more appropriate to talk of *validity* instead of *truth*, but for the present purposes this is not essential.

false, or non-significant, respectively.¹⁹ It is clear that the generalised first-order framework given by this semantic valuation rule and the type of model specified in the preceding section does allow for truth-value gaps. Moreover, as no general assumptions concerning the set-theoretic relations between the positive and the negative components of predicate denotations have been made in section 2.2, this generalised framework also allows for the following two (relativised) classifications of predicates: if $P \in \mathbf{P}_i$ and $M \in \mathbf{M}$ then

- (i) P is *consistent* in M iff_{df} for all $\sigma \in M$, $x^{(k_1)}, \dots, x^{(k_i)} \in \mathbf{Var}$
 $v(\langle P(x^{(k_1)}, \dots, x^{(k_i)}), M, \sigma \rangle, T)$ iff not- $v(\langle P(x^{(k_1)}, \dots, x^{(k_i)}), M, \sigma \rangle, F)$;
- ii) P is *complete* in M iff_{df} for all $\sigma \in M$, $x^{(k_1)}, \dots, x^{(k_i)} \in \mathbf{Var}$
 $v(\langle P(x^{(k_1)}, \dots, x^{(k_i)}), M, \sigma \rangle, T)$ or $v(\langle P(x^{(k_1)}, \dots, x^{(k_i)}), M, \sigma \rangle, F)$

2.4. The Classical Framework

It is easy to see that the framework just introduced is indeed – as suggested by our nomenclature – a generalisation of the classical one in the sense that for any \mathbf{M} , there is a proper sub-class \mathbf{M}^{cl} of \mathbf{M} – namely the one given by

$$M \in \mathbf{M}^{cl} \text{ iff}_{df} \text{ for all } P \in \mathbf{P}_k \text{ and all } k: \mathbf{non-P}^M = \mathbf{M} \setminus \mathbf{P}^M$$

– which is isomorphic (with respect to semantic valuations) to the set of classically conceived models for \mathbf{B} based on \mathbf{M} . To see this we only need to consider the fact that our semantic valuation relation v , restricted to \mathbf{M}^{cl} , is actually a *function* which conforms to the classical rule that (for all $\sigma \in M \in \mathbf{M}^{cl}$)

$$v(P(x^{(k_1)}, \dots, x^{(k_i)}), M, \sigma) = \begin{cases} T & \text{if } \langle \sigma(x^{(k_1)}), \dots, \sigma(x^{(k_i)}) \rangle \in \mathbf{P}^M \\ F & \text{else} \end{cases}$$

One obvious advantage of this is that it allows us to discuss the classical framework in terms of \mathbf{M}^{cl} .²⁰

3. The Semantics of Numerical Identity

We can discuss the envisaged generalisation of the classical theory of identity, without loss of generality but with a considerable reduction of notational complexity, if we restrict the predicate scheme of our purely denota-

¹⁹ Note that the non-significance of “ $P(x)$ ” represented by $v(\langle P(x), M, \sigma \rangle, N)$ is not due to some referential failure of “ x ”, but is meant to reflect the inapplicability of P to the individual referred to by “ x ” under the relevant interpretation σ .

²⁰ Take, for example, the fact that since all $P \in \mathbf{P}$ are consistent and complete in all $M \in \mathbf{M}^{cl}$, both consistency and completeness are genuine classifications only in the generalised framework, and not in the classical one.

tional languages to *Monadic* and *Identity Predicate* constants, i.e. by considering only vocabularies of the form:

$$(MIP) \quad P_1 \dot{\cup} Id \dot{\cup} Var \quad (\text{with } Id = \{Id_1, \dots, Id_n\} \subseteq P_2).$$

3.1. The Classical Theory of Identity

Classically, identity predicates are interpreted to be the set-theoretical identity $=_e$ between the elements of the (classical) domain of the (classical) model in question. Given our representation of the classical framework in the atomistic generalised framework, this means that, if $Id_M^{cl} \subseteq M_B^{cl}$ is given by

$$M \in Id_M^{cl} \text{ iff } Id_k^M = \{\langle x, y \rangle \in M \times M : x =_e y\} \text{ (for all } k)$$

then Id_1, \dots, Id_n – as interpreted in some $M \in M_B^{cl}$ – will, in the classical conception, be identity predicates iff $M \in Id_M^{cl}$, i.e. iff they are interpreted as *the* identity relation. This specification of what it is to be an identity predicate obviously reflects the classical use of “identity relation” as a proper name. In order to explicate the conception of such relations put forward in the introductory section, I shall now introduce what might be called the *classical (semantic²¹) theory of identity*.

The classical theory of identity (for first-order language based on an MIP-vocabulary B) is given i) by the following system of *Classical Identity Rules*, which stipulate, for any given classical model M for B , that (for all $P \in P_1$, $Id_k \in Id$, $x, y \in Var$, $\sigma \in M$)

- (CIR.1) $v(Id_k(x, x), M, \sigma) = T$ (Reflexivity);
- (CIR.2) If $v(Id_k(x, y), M, \sigma) = T$ then $v(Id_k(y, x), M, \sigma) = T$ (Symmetry);
- (CIR.3) If $v(Id_k(x, y), M, \sigma) = T$ and $v(Id_k(y, z), M, \sigma) = T$ then $v(Id_k(x, z), M, \sigma) = T$ (Transitivity);
- (CIR.4) If $v(Id_k(x, y), M, \sigma) = T$ and $v(P(x), M, \sigma) = T$ then $v(P(y), M, \sigma) = T$ (Substitution);

and ii) the condition that

- (C) for all MIP-vocabularies B' (with $Id' = Id$), the classical identity rules hold for all B' -models $M' \in M_{B'}$ (i.e. based on the same M as M) with the same interpretation of Id_1, \dots, Id_n as M .

²¹ My use of “semantic” here is to introduce a distinction to the axiomatic/deductive conception of a theory.

It is easy to see that all the models of \mathbf{Id}_M^{cl} satisfy the classical identity rules. What is crucial, however, is that this semantic theory manages to identify the classical identity predicates – in the sense that

$$M \text{ satisfies (C) iff } M \in \mathbf{Id}_M^{cl}$$

– without reference to a particular meta-language relation, for only this sort of characterisation can be generalised in the way envisaged in the introductory section.

3.2. The Generalised Theory of Identity

After this discussion of the classical theory of identity, let us now turn to the generalisation of the conception of numerical as envisaged at the end of the introductory section. Our aim now is thus to find a theory of identity (for first-order languages based on a MIP-vocabulary \mathbf{B}) which allows, in particular, for truth-valueless elementary identity sentences. The semantic theory (in the above sense), which I suggest satisfies this, is given i) by the following system of *Generalised Identity Rules*,²² which stipulate, for any given atomistic generalised model M for \mathbf{B} , that (for all $P \in \mathbf{P}_1$, $Id_k, Id_m \in \mathbf{Id}$, $x, y \in \mathbf{Var}$, $\sigma \in M$)

(GIR.1) If $\text{not-}v(\langle Id_k(x, x), M, \sigma \rangle, N)$ and $\text{not-}v(\langle Id_m(x, x), M, \sigma \rangle, N)$ for some $x \in \mathbf{Var}$, $\sigma \in M$, then for all $y \in \mathbf{Var}$, $\sigma' \in M$:

$\text{not-}v(\langle Id_k(y, y), M, \sigma' \rangle, N)$ iff $\text{not-}v(\langle Id_m(y, y), M, \sigma' \rangle, N)$.

(GIR.2) If $\text{not-}v(\langle Id_k(x, y), M, \sigma \rangle, N)$

then $\text{not-}v(\langle Id_k(x, x), M, \sigma \rangle, N)$ and $\text{not-}v(\langle Id_k(y, y), M, \sigma \rangle, N)$.

(GIR.3) If $\text{not-}v(\langle Id_k(x, x), M, \sigma \rangle, N)$

then $v(\langle Id_k(x, x), M, \sigma \rangle, T)$ (Generalised Reflexivity).

(GIR.4) If $v(\langle Id_k(x, y), M, \sigma \rangle, V)$

then $v(\langle Id_k(y, x), M, \sigma \rangle, V)$ (Generalised Symmetry)

– where V stands for any of the three semantic values considered here.

(GIR.5) If $v(\langle Id_k(x, y), M, \sigma \rangle, T)$ and $v(\langle Id_k(y, z), M, \sigma \rangle, T)$

then $v(\langle Id_k(x, z), M, \sigma \rangle, T)$ (Transitivity).

(GIR.6) If $v(\langle Id_k(x, y), M, \sigma \rangle, T)$ and $v(\langle P_i(x), M, \sigma \rangle, V)$

then $v(\langle P_i(y), M, \sigma \rangle, V)$ (Generalised Substitution).

²² Let me point out that by such an ‘identity rule’ I mean a rule which is part of the specification of what it is for a predicate to be an *identity* predicate – which should not be confused with what I earlier referred to as ‘*identifying* rules’ linked with the application of identity predicates.

and ii) the condition that

- (G) for all MIP-vocabularies B' (with $Id' = Id$) the generalised identity rules hold for all B' -models $M' \in \mathbf{M}_{B'}$ (i.e. based on the same \mathbf{M} as M) with the same interpretation of Id_1, \dots, Id_n as M .

The first two of these rules concern the distribution of ‘truth-value gaps’ mentioned in the introductory section. They are motivated by what we called the criteria-linked conception of identity predicates. Thus (GIR.1) is meant to reflect the idea that if two sets of criteria are *both* applicable to some individual, then the features required for an application by either set of criteria must be related in such a way that the first set will *always* be applicable if and only if the second one is. The motivation behind the other five rules is, I believe, self-evident.²³ The condition (G), finally, is nothing but the appropriate generalisation (C) used to characterise the classical theory of identity.

Given this generalised theory of identity – i.e., to be more precise, the ‘atomistic generalised semantic theory of identity’ (given that it was formulated in terms of the atomistic generalised semantic framework) – which, I take it, reflects the criteria-linked conception of numerical identity discussed in the introductory section, we are now in a position to search for a semantic framework for languages with predicates that refer to the sort of relations characterised by this theory.

3.3. The Atomistic Generalised Framework for Languages with Identity

The classical semantic framework for languages with identity clearly does not adequately reflect the conception of identity given in the generalised theory. But it is not all too difficult to transform (our representation of) this classical framework into a model-theoretic system which will at least be able to represent the variety of identity predicates allowed for in the generalised conception. All we need to do is i) to introduce for every set \mathbf{M} used in the construction of our atomistic generalised models a function.

$$\alpha: Id \rightarrow 2^{\mathbf{M}} \setminus \{\emptyset\}$$

which transforms \mathbf{M} into a partitioned set \mathbf{M}^α ,²⁴ ii) to restrict the semantic framework to those atomistic generalised models for B (collectively referred

²³ It is easy to see that every classical model which satisfies the classical theory of identity will also satisfy this generalised theory. Indeed, if we restrict the scope of models to \mathbf{M}^{cl} , then the two theories turn out to be equivalent, in the sense that any $M \in \mathbf{M}^{cl}$ satisfies CIR iff it satisfies GIR.

²⁴ α is thus meant to satisfy the conditions that

(i) $\alpha(Id_k) \cap \alpha(Id_m) \neq \emptyset$ iff $\alpha(Id_k) = \alpha(Id_m)$ and (ii) $\mathbf{M} = \bigcup_{k=1}^n \alpha(Id_k)$.

to as \mathbf{M}_B^α) in which an Id_k pertains to all and only the individuals of $\alpha(k)$,²⁵ and iii) to stipulate that (for $M \in \mathbf{M}_B^\alpha$)

$$M \in \mathbf{Id}_{M^\alpha} \text{ iff }_{df} \text{ for all } Id_k \in \mathbf{Id} \\ \mathbf{Id}_k^M = \{\langle x, y \rangle \in \alpha(k) \times \alpha(k) : x =_\epsilon y\} \text{ and } \mathbf{non-Id}_k^M \text{ is a symmetric} \\ \text{sub-set of } \alpha(k) \times \alpha(k).$$

It is easy to see, not only that this framework generalises the one discussed in section 3.1,²⁶ and that all the models of \mathbf{Id}_{M^α} satisfy the generalised identity rules, but also that

$$M \in \mathbf{Id}_{M^\alpha} \text{ iff } M \text{ satisfies the condition (G) of the generalised theory of identity,}^{27}$$

which justifies the conclusion that

- (1) Id_1, \dots, Id_n – as interpreted in the models of \mathbf{Id}_{M^α} – are identity predicates (in the sense of the generalised theory of identity).²⁸

The fact then that

- (2) $v(\langle Id_k(x, x), M, \sigma \rangle, N)$ iff $\sigma(x) \notin \alpha(Id_k)$ (for all $Id_k \in \mathbf{Id}, M \in \mathbf{Id}_{M^\alpha}$)

and that

- (3) any identity predicate can be *inconsistent* and/or *incomplete*

not only demonstrates that there can be truth-valueless elementary identity sentences in this framework (3), even if they are of the reflexive form (2), but also justifies an interpretation of α as an assignment of domains to the identity predicates in question. Accordingly I shall assume that the way to interpret the notion of the ‘domain of an identity predicate’ Id_k (interpreted in $M \in \mathbf{Id}_{M^\alpha}$) is to specify that $dom_M^\alpha(Id_k) =_{df} \alpha(Id_k)$.²⁹ Moreover, given that (for any $M \in \mathbf{Id}_{M^\alpha}$)

- (4) if $dom_M^\alpha(Id_k) \neq dom_M^\alpha(Id_m)$ then $Id_k \not\equiv^M Id_m$ (i.e. Id_k and Id_m have different denotations in M),

²⁵ In other words:

$M \in \mathbf{M}_B^\alpha$ iff $_{df} \alpha(k) = \{a \in \mathbf{M} : (\exists b \in \mathbf{M}) (\langle a, b \rangle \in Id_k|_M \vee \langle b, a \rangle \in Id_k|_M)\}$
 $- Id_k|_M$ stands for the significance range $\mathbf{Id}_k^M \cup \mathbf{non-Id}_k^M$ of Id_k in M .

²⁶ In the sense that if $\alpha(Id_k) = \mathbf{M}$ (for all Id_k) then \mathbf{Id}_k^M is a (proper) sub-set of \mathbf{Id}_{M^α} , namely the one given by the condition that $\mathbf{non-Id}_k^M = \mathbf{M} \times \mathbf{M} \setminus \mathbf{Id}_k^M$.

²⁷ The assumption is, of course, that this theory be given in terms of \mathbf{M}_B^α (instead of \mathbf{M}_B).

²⁸ The first two generalised identity rules, for example, are satisfied, because \mathbf{M}^α is a partitioned set, while (GIR.6) is satisfied because of the fact that (for all $\sigma \in M \in \mathbf{Id}_{M^\alpha}$ and all $Id_k \in \mathbf{Id}$) $v(\langle Id_k(x, y), M, \sigma \rangle, T)$ iff $\sigma(x) =_\epsilon \sigma(y)$.

²⁹ Note that hence for all $M, M' \in \mathbf{Id}_{M^\alpha}$ $dom_M^\alpha(Id_k) = dom_{M'}^\alpha(Id_k)$

it will also be clear that the framework put forward here mirrors yet another feature of our original conception, in that it allows for a genuine plurality of identity predicates, provided that we adopt the principle that (for any $M \in \mathbf{Id}_M^\alpha$)

(SYN) if Id_k and Id_m are synonymous, then $Id_k \stackrel{M}{\neq} Id_m$.

So far so good. The fact, however, that the converse of (4) is not true runs counter to the ontologically generative aspect of the identity conception put forward in the introductory section, for if a difference in meaning between two identity predicates Id_k and Id_m is actually reflected in their denotations in some model M (i.e. if $Id_k \neq Id_m$), then this should likewise be reflected in the domains they generate in this model (i.e. then $dom_M^\alpha(Id_k) \neq dom_M^\alpha(Id_m)$). Moreover, if – in keeping with this original conception – we define that (for $Id_k, Id_m \in \mathbf{Id}, M \in \mathbf{Id}_M^\alpha$)

(CAT) Id_k and Id_m are of the *same category* in M iff_{df}
 for all $\sigma \in M, x \in \mathbf{Var}$
 $not\text{-}v(Id_k(x, x), M, \sigma), N$ iff $not\text{-}v(Id_m(x, x), M, \sigma), N$

then it follows from (2) that

(5) Id_k and Id_m are of the same category in M iff $dom_M^\alpha(Id_k) = dom_M^\alpha(Id_m)$,

which also runs counter to this ontologically generative aspect: the relation given in (CAT) formally represents the fact the criteria associated with the identity predicates in question make reference to the same sorts of features; this, however, does by no means imply that the criteria themselves (and thus the generated domains) need be the same in both cases. Yet none of this should be surprising, for even though the framework presented here does, as it were, capture a (semantic) aspect of the initially introduced conception of identity predicates, it treats the link between these predicates and their domains in precisely the same fashion as the classical semantics. In the next section, my aim will now be to specify a formal semantic framework which is able to reflect both these aspects of our conception.

4. The Structurally Enriched Generalised Framework

4.1. The Sub-Ontological Structure of Models

The models of the generalised framework introduced in section 2 were what I called ‘atomistic’ in the sense that their ultimate building blocks were

seen as unstructured (object-language) individuals.³⁰ In order to incorporate the ontologically generative nature of object-language identity predicates, I shall now enrich the structure of such an atomistic generalised model to achieve a representation of a 'sub-ontological' or 'internal' structure of object-language individuals. The idea is that (object-language) identity predicates are ontologically creative insofar as they are taken to govern the 'generation', or 'construction' of *structured* individuals out of certain units which I shall refer to as 'generators', and that the structure of these individuals reflects the basic identifying acts involved in their construction.³¹ To put it differently, the conception of the ontologically generative nature of identity predicates which I intend to represent here is that identity predicates govern certain basic identifying acts (expressable by elementary identity sentences) which can be interpreted as generating, or constructing, complex structured entities out of a manifold of generators, entities which subsequently serve as the individuals of the object-language in question.

Given the discussion in the preceding section, it should not be surprising that, in order to achieve a formal representation of this conception, I will not be looking at yet another version of the atomistic generalised models discussed in section 2.2, but rather focus on trying to modify the atomistic building blocks of these models. Instead of some class representing a collection of object-language individuals, the construction of a model of the *structurally enriched* framework which I am about to introduce is thus based on a set (from the realm of set-theory), say C , which is meant to represent such a manifold of generators. In a first constructive phase, this class is taken to be structured by an '*elementary identifying function*'

³⁰ This meant, in particular, that the set-theoretical meta-language employed in the characterisation of the generalised semantic framework dealt only with individuals of one kind (and, of course, with set-theoretical constructs based on them) and thus involved only one (meta-language) identity predicate (denoted by $=_{\epsilon}$).

³¹ No identifying act can be performed in the absence of all structure in the 'manifold of reference', i.e. the manifold on the basis of which the identifying acts are performed. Yet this does not mean that the structure required is that of a full-blown identity structure. All that is minimally required is that the reference manifold display certain units for the identifying acts to 'latch on to'. These units can be individuals (i.e. entities which are already subject to some identity relation) or 'mere' units. The paradigm of such mere units I have in mind is the kind of units we are presented with in sensory experience, i.e. the presentations of the sort of spatially cohesive units inherent in visually apprehended shapes, contours and lines. A more abstract example would be the sort of cohesive units introduced into a class by means of an equivalence relation (see section 5). (Note, incidentally, that even though such a manifold of equivalence units is most naturally transformed into an identity structure by means of the set-theoretical identity, this is by no means the only way in which this can be achieved.)

$$(I) \quad \delta: \begin{cases} \text{Var} \rightarrow \mathbf{C} \\ \text{Id} \rightarrow 2^{\mathbf{C} \times \mathbf{C}} \times 2^{\mathbf{C} \times \mathbf{C}} \end{cases}$$

satisfying the following three conditions: if \mathbf{Id}_k^δ and $\mathbf{non-Id}_k^\delta$ denote the first and the second component of $\delta(\text{Id}_k)$, respectively, and $\mathbf{C}^\delta[\text{Id}_k] =_{df} \{c \in \mathbf{C} : \langle c, c \rangle \in \mathbf{Id}_k^\delta\}$, then

- i) $\mathbf{Id}_k^\delta \subseteq \mathbf{C} \times \mathbf{C}$ is an equivalence relation on $\mathbf{C}^\delta[\text{Id}_k]$,
- ii) $\mathbf{non-Id}_k^\delta \subseteq \mathbf{C} \times \mathbf{C}$ is an symmetric relation on $\mathbf{C}^\delta[\text{Id}_k]$,
- iii) $\{\mathbf{C}^\delta[\text{Id}_k] : k = 1, \dots, n\}$ is a partition of \mathbf{C}

(let me use “ \mathbf{C}^δ ” to denote the resulting *structured* class). As suggested by the nomenclature, \mathbf{Id}_k^δ and $\mathbf{non-Id}_k^\delta$ are meant to reflect the results of the above-mentioned basic Id_k -identifying acts, and it is thus easy to see why δ is meant to conform to i) and ii).³² The third condition is introduced to reflect the categoricity inherent in the suggested informal conception of identity predicates, and, as such, it forms the semantic counterpart to the first of the above-introduced general identity rules. In other words, the partition of \mathbf{C} is meant to reflect a classification of the generators represented in \mathbf{C} according to the sort of features required for an application of the identity predicates in question³³ (accordingly, $\mathbf{C}^\delta[\text{Id}_k]$ will be called the *categorical signature* of Id_k in \mathbf{C}^δ). Before turning to the construction of structured individuals, let me use the following figure to illustrate (for $\text{Id} = \{\text{Id}_1, \text{Id}_2, \text{Id}_3\}$) the sort of structures introduced in this section:

³² Having said this I must add a note of caution: even though our characterisation of structurally enriched models will use \mathbf{Id}_k^δ and $\mathbf{non-Id}_k^\delta$ as auxiliary structures in constructing Id_k -individuals, it is these individuals, and not auxiliary constructs, which represent the products of the basic identifying acts.

³³ I must point out that this sort of representation is not adequate for identity predicates which make reference to a plurality of essential identifying features, but this shortcoming will not impinge in an essential way on the explicatory force of the framework to be developed, and, moreover, could be overcome by introducing some (partial) order into our representation of the structured reference manifold.

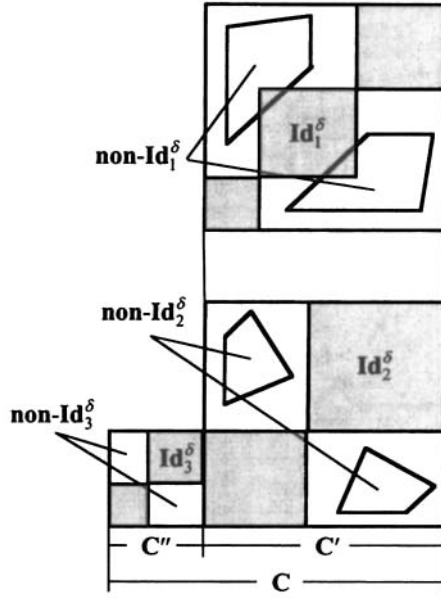


Figure 1

Note that, in this case, Id_1 and Id_2 have the same categorial signature $C^\delta[Id_1] = C^\delta[Id_2] = C'$ (distinct from $C^\delta[Id_3] = C''$), and yet, as we shall presently see, they are distinct identity predicates – indeed Id_1 turns out to be inconsistent and incomplete, while Id_2 is consistent and incomplete. Only Id_3 is classical in the sense of both being complete and consistent.

4.2. Structured Individuals

The type of structured classes (C^δ) specified in the preceding section allows us to construct the envisaged formal representations of structured individuals generated by means of identity predicates, which subsequently will collectively replace the sets \mathbf{M} of atomic individuals used in the atomistic characterisation of generalised models given in section 2.2. This is accomplished in two stages. In the first stage we collect, for any given c of the categorial signature of the identity predicate we are considering (say Id_k), on the one hand, all the instances of basic identities with c (graphically represented in Figure 2 by the $C^\delta[Id_k]$ -elements linked by a line-segment), and, on the other, all the instances of basic non-identities with c (represented by means of double-arrows).

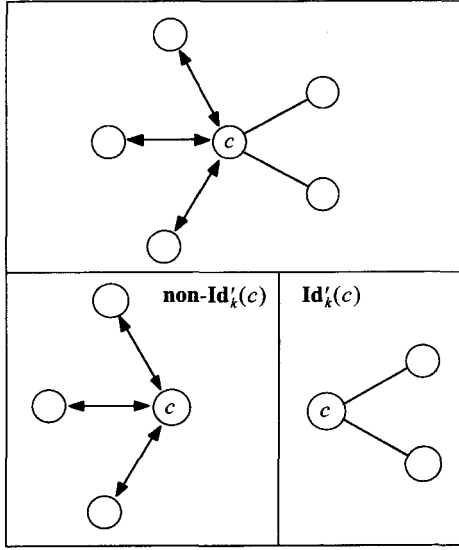


Figure 2

That is to say, for every $c \in \mathbf{C}^\delta[Id_k]$ we introduce two further auxiliary set-theoretical constructs:

$$\begin{aligned} \mathbf{Id}_k'(c) &=_{df} \{ \langle a, b \rangle \in \mathbf{Id}_k^\delta : \langle a, c \rangle \in \mathbf{Id}_k^\delta \text{ or } \langle c, b \rangle \in \mathbf{Id}_k^\delta \} \\ \mathbf{non-Id}_k'(c) &=_{df} \{ \langle a, b \rangle \in \mathbf{non-Id}_k^\delta : \langle a, c \rangle \in \mathbf{non-Id}_k^\delta \text{ or } \langle c, b \rangle \in \mathbf{non-Id}_k^\delta \} \end{aligned}$$

In the second stage we then simply unify these two constructs with the relevant constructs for all the elements of the \mathbf{Id}_k^δ -equivalence class $[c]_k$ of c (see Figure 3), i.e. we form the constructs

$$\mathbf{Id}_k(c) =_{df} \bigcup_{x \in [c]_k} \mathbf{Id}_k'(x), \text{ and } \mathbf{non-Id}_k(c) =_{df} \bigcup_{x \in [c]_k} \mathbf{non-Id}_k'(x)$$

which, when conjoined to form the par $\langle \mathbf{Id}_k(c), \mathbf{non-Id}_k(c) \rangle$, represent the structured individual arising from the generator (represented by) c through an application of Id_k .³⁴

³⁴ As it happens $\mathbf{Id}_k(c) = [c]_k \times [c]_k$ and $\mathbf{non-Id}_k(c) = \mathbf{non-Id}_k^\delta \times E_k(c)$ (with $E_k(c) =_{df} ([c]_k \times \mathbf{C}^\delta[Id_k]) \cup (\mathbf{C}^\delta[Id_k] \times [c]_k)$). Yet this characterisation, although useful for certain purposes, is less illustrative as concerns the internal structure of these individuals than the initial characterisation.

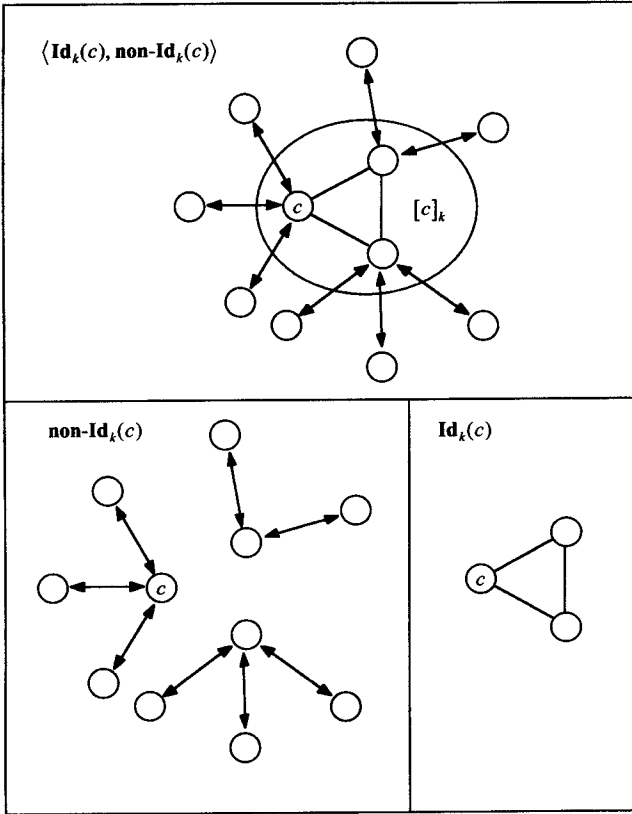


Figure 3

In section 5.2, I shall discuss in some detail this sort of generation in the context of generating (Cauchy-) real numbers from the space of (simple) infinite sequences of rational numbers as manifold of generators. At this stage it might be useful to think of the difference between the atomistic conception of individuals and the one introduced here as being analogous to the difference between the atomistic and the relational conceptions of points in space.

4.3. Structurally Enriched Generalised Conception of Formal Models

Given a particular (representation of a) structured reference manifold \mathbf{C}^δ , we thus have, for every c in the relevant categorial signature $\mathbf{C}^\delta[\text{Id}_k]$ (of an identity predicate Id_k)³⁵ a set-theoretic construct, namely $\langle \text{Id}_k(c), \text{non-}$

³⁵ $\mathbf{C}^\delta[\text{Id}_k]$, we may recall, is meant to represent those generators which display the features required for an application of Id_k .

$\text{Id}_k(c)$), the structure of which is determined by the basic identifying acts (carried out by means of Id_k) involving c , and which, as such, is taken to represent an Id_k -individual. It should thus not be surprising that the domain of discourse determined by the rules governing the use of Id_k be represented by

$$\mathbf{D}_k^\delta =_{df} \{ \langle \text{Id}_k(c), \text{non-Id}_k(c) \rangle : c \in \mathbf{C}^\delta[\text{Id}_k] \} \subseteq 2^{\mathbf{C}^\delta[\text{Id}_k] \times \mathbf{C}^\delta[\text{Id}_k]} \times 2^{\mathbf{C}^\delta[\text{Id}_k] \times \mathbf{C}^\delta[\text{Id}_k]}$$

and that the domains given in the partitioned set \mathbf{M}^a used in the atomistic characterisation of our generalised formal models (see section 3.3) be replaced by these constructed domains. The characterisation of the structurally enriched generalised formal models (based on such a structured reference manifold \mathbf{C}^δ) is thus to be given in terms of the class $\Sigma_{\mathbf{C}^\delta}$ of “formal \mathbf{C}^δ -interpretations”, which are functions of the following type:

$$(II) \quad \sigma : \begin{cases} \text{Var} \rightarrow \bigcup_{k=1..n} \mathbf{D}_k^\delta \\ \mathbf{P}_1 \rightarrow \bigcup_{k=1..n} (2^{\mathbf{D}_k^\delta} \times 2^{\mathbf{D}_k^\delta}) \\ \text{Id} \rightarrow \bigcup_{k=1..n} (2^{\mathbf{D}_k^\delta \times \mathbf{D}_k^\delta} \times 2^{\mathbf{D}_k^\delta \times \mathbf{D}_k^\delta}) \end{cases}$$

By itself, there is not much difference between this and the characterisation of formal interpretations given in section 2.2. This, however, changes drastically if one admits the possibility that, at least in some contexts, the interpretation of variables is to be executed by using the generations underlying our construction, i.e. if we stipulate that (for every $\sigma \in \Sigma_{\mathbf{C}^\delta}$ and every $x \in \text{Var}$) $\sigma(x)$ must be specifiable as

$$(III) \quad \sigma(x) = \langle \text{Id}_k(\delta'(x)), \text{non-Id}_k(\delta'(x)) \rangle,$$

for some δ' constructively equivalent to δ ,³⁶ for this sort of assignment is only taken to be well-defined if $\delta'(x)$ is in the relevant categorial signature of Id_k . This means that by adopting (III), the proposed enriched semantic framework will – in stark contrast to all the semantic frameworks considered thus far – *necessitate* the adoption of a genuinely *many-sorted* linguistic framework, in that we are forced to distinguish between kinds of variables, according to the identity predicate they are associated with, which amounts to partitioning Var into disjoint classes $\text{Var}[\text{Id}_1], \dots, \text{Var}[\text{Id}_n]$ by stipulating that $x \in \text{Var}[\text{Id}_k]$ iff $\delta(x) \in \mathbf{C}^\delta[\text{Id}_k]$. This sort of classification of variables, incidentally, ensures (in

³⁶ Two functions δ and δ' of the type specified under (I) are said to be constructively equivalent (for a given class \mathbf{C}) if $\delta' \text{Id} = \delta \text{Id}$. Note that two such functions have the same categorial signatures in \mathbf{C} and define the same domains, and that for any individual a of a domain \mathbf{D}_k^δ (specified in terms of δ) and any $x \in \text{Var}$ with $\delta(x) \in \mathbf{C}^\delta[\text{Id}_k]$ there is a δ' , constructively equivalent to δ , with $a =_e \langle \text{Id}_k(\delta'(x)), \text{non-Id}_k(\delta'(x)) \rangle$.

keeping with the underlying intuition) that the interpretation of any variable is confined to one domain, i.e. that $\sigma(\text{Var}[Id_k]) \subseteq \mathbf{D}_k^\delta$ (for every $\sigma \in \Sigma_{\mathbf{C}^\delta}$ which satisfies (II)).

In keeping with this intuition, I shall assume that “formal \mathbf{C}^δ -interpretations” are confined in this manner, not only for variables, but also for predicates. In other words, I shall assume that $\Sigma_{\mathbf{C}^\delta}$ is a class of functions of type (I) which is maximal with respect to the following three conditions:

- i) for any $x \in \text{Var}$ there is exactly one domain \mathbf{D}_k^δ s.t. $\sigma(x) \in \mathbf{D}_k^\delta$ for all $\sigma \in \Sigma_{\mathbf{C}^\delta}$,
- ii) for any $P \in \mathbf{P}_1$ there is exactly one domain \mathbf{D}_k^δ s.t. $\sigma(P) \in 2^{\mathbf{D}_k^\delta \times \mathbf{D}_k^\delta}$ for all $\sigma \in \Sigma_{\mathbf{C}^\delta}$, and
- iii) $\sigma(Id_k) \in 2^{\mathbf{D}_k^\delta \times \mathbf{D}_k^\delta} \times 2^{\mathbf{D}_k^\delta \times \mathbf{D}_k^\delta}$ for any $Id_k \in \mathbf{Id}$.

If now, in analogy to section 2.2, a (structurally enriched generalised formal) model \mathbf{M} for an MIP-vocabulary \mathbf{B} (based on \mathbf{C}^δ) is defined to be any sub-set of $\Sigma_{\mathbf{C}^\delta}$ which is maximal with respect to the condition that

$$\sigma(X) = \sigma'(X) \text{ for all } \sigma, \sigma' \in \mathbf{M} \text{ and all } X \in \mathbf{P}_1 \cup \mathbf{Id}$$

– the totality of such models shall be denoted by \mathbf{C}_B^δ (or \mathbf{C}^δ) – then it will not be difficult to see that the effect of constraining formal \mathbf{C}^δ -interpretations in this manner is simply that i) all these models will have the same domains, ii) a variable will always range over one and the same domain, iii) a monadic predicate will always pertain to the individuals of one and the same domain, and iv) any identity predicate will always pertain to the individuals which it served to construct – which collectively will obviously have to be the domain of the identity predicate in question, i.e.

$$\text{dom}_M^\delta(Id_k) =_{df} \mathbf{D}_k^\delta \text{ (for all } \mathbf{M} \in \mathbf{C}_B^\delta \text{)}.$$

Within the framework given by this conception of structurally enriched models (together with the generalised valuation rules introduced in section 2.3) we are now in a position to define a formal semantic framework which reflects both the purely semantic and the ontologically generative aspects of identity predicates and which provides a formal explication of the sort of essential disjointness discussed in the introductory section. To do this, we first specify for each $Id_k \in \mathbf{Id}$ the denotation $\langle \mathbf{Id}_k^M, \text{non-Id}_k^M \rangle$ in a structurally enriched model $\mathbf{M} \in \mathbf{C}^\delta$ by stipulating that for $c_1, c_2 \in \mathbf{C}^\delta[Id_k]$ ($k = 1, \dots, n$)

$$\begin{aligned} \langle \mathbf{Id}_k(c_1), \text{non-Id}_k(c_1) \rangle, \langle \mathbf{Id}_k(c_2), \text{non-Id}_k(c_2) \rangle \rangle &\in \mathbf{Id}_k^M \text{ iff }_{df} \langle c_1, c_2 \rangle \in \mathbf{Id}_k(c_1) \cap \mathbf{Id}_k(c_2) \\ \langle \mathbf{Id}_k(c_1), \text{non-Id}_k(c_1) \rangle, \langle \mathbf{Id}_k(c_2), \text{non-Id}_k(c_2) \rangle \rangle &\in \text{non-Id}_k^M \text{ iff }_{df} \\ \langle c_1, c_2 \rangle &\in \text{non-Id}_k(c_1) \cap \text{non-Id}_k(c_2) \end{aligned}$$

and by considering only those enriched models which conform to this definition³⁷ (which shall be collectively denoted by \mathbf{Id}_{C^δ}). As in the case of our atomistic generalised framework (see section 3.3) we will now find, not only that

$M \in \mathbf{Id}_{C^\delta}$ iff M satisfies the condition (G) of the generalised theory of identity,³⁸

and thus that

- (1) Id_1, \dots, Id_n – as interpreted in the models of \mathbf{Id}_{C^δ} – are identity predicates (in the sense of the generalised theory of identity),³⁹

but also that (for all $Id_k \in \mathbf{Id}$, $M \in \mathbf{Id}_{C^\delta}$)

- (2) $\forall \langle \langle Id_k(x, x), M, \sigma \rangle, N \rangle$ iff $\sigma(x) \notin \text{dom}_M^\delta(Id_k)$;
 (3) identity predicates can be *inconsistent* and/or *incomplete*;⁴⁰
 (4) if $\text{dom}_M^\delta(Id_k) \neq \text{dom}_M^\delta(Id_m)$ then $Id_k \not\equiv^M Id_m$.

That is to say, our enriched semantic framework reflects the semantic aspects of our initial informal conception of (ontologically generative) identity predicates. Yet it manages to avoid the representational shortcomings of the atomistic generalised framework discussed at the end section 3.3, in that we also have that (for all $Id_k, Id_m \in \mathbf{Id}$, $M \in \mathbf{Id}_{C^\delta}$)

- (5) $Id_k \equiv^M Id_m$ iff $\text{dom}_M^\delta(Id_k) = \text{dom}_M^\delta(Id_m)$,
 (6) Id_k and Id_m are of the same category in M iff $C^\delta[Id_k] = C^\delta[Id_m]$,
 (7) if $\text{dom}_M^\delta(Id_k) \cap \text{dom}_M^\delta(Id_m) \neq \emptyset$ then $C^\delta[Id_k] = C^\delta[Id_m]$ (but *not* the converse).

³⁷ This definition is based on the the following three facts:

i) $\langle \mathbf{Id}_k^M, \mathbf{non-Id}_k^M \rangle \in 2^{D_k^\delta \times D_k^\delta} \times 2^{D_k^\delta \times D_k^\delta}$
 ii) for every $a \in D_k^\delta$ there is (by construction) a $c \in C^\delta[Id_k]$ such that $a =_\in \langle \mathbf{Id}_k(c), \mathbf{non-Id}_k(c) \rangle$ (with “ $=_\in$ ” denoting the meta-linguistic set-theoretical identity relation); and
 iii) if $\langle \mathbf{Id}_k(c_1), \mathbf{non-Id}_k(c_1) \rangle =_\in \langle \mathbf{Id}_k(c_2), \mathbf{non-Id}_k(c_2) \rangle$ then

a) $\langle c_1, c \rangle \in \mathbf{Id}_k(c_1) \cap \mathbf{Id}_k(c)$ iff $\langle c_2, c \rangle \in \mathbf{Id}_k(c_2, c) \in \mathbf{Id}_k(c_1) \cap \mathbf{Id}_k(c)$; and
 b) $\langle c_1, c \rangle \in \mathbf{non-Id}_k(c_1) \cap \mathbf{non-Id}_k(c)$ iff $\langle c_2, c \rangle \in \mathbf{non-Id}_k(c_1) \cap \mathbf{non-Id}_k(c)$.

³⁸ This time, I am referring to a reformulation of this theory in terms of \mathbf{C}_g^δ .

³⁹ (GIR.6) is again satisfied because of the fact that (for all $\sigma \in M \in \mathbf{Id}_{C^\delta}$, $Id_k \in \mathbf{Id}$)
 $\forall \langle \langle Id_k(x, y), M, \sigma \rangle, T \rangle$ iff $\sigma(x) =_\in \sigma(y)$

⁴⁰ This, and indeed our classification of the identity predicates graphically represented in Figure 1, follows directly from the fact that for any model $M \in \mathbf{C}^\delta$

Id_k is consistent in M iff $\mathbf{Id}_k^\delta \cap \mathbf{non-Id}_k^\delta = \emptyset$,

Id_k is complete in M iff $\mathbf{Id}_k^\delta \cup \mathbf{non-Id}_k^\delta = C^\delta[Id_k] \times C^\delta[Id_k]$.

As far as an explication of the possibility of essentially disjoint domains is concerned, the framework presented here shows that, as long as a unification of two domains involves the creation of a joint domain, then any two domains generated by identity predicates of different categories will be essentially disjoint (i.e. *not* unifiable): it is always possible to introduce new identity predicates (and thus new domains) into our structurally enriched models by expanding the relevant elementary identifying function (δ). Such an expansion is, however, only admissible, if the categorial signatures of the new predicates respect the categorisation of the old ones.

5. *Non-Significance versus Non-Existence*

Let me recapitulate some of the ideas put forward in this paper by comparing them with Dana Scott's point of view as presented in his "Identity and Existence in Intuitionistic Logic" [1979], which, although similar in some respects, turns out to be fundamentally different from the one advocated in this essay.

5.1. *Scott's Conception of Identity and Existence*

The very first paragraph in section 1 of Scott's paper – where he accepts that, for constant names " a " and " b ", the sentence " $a = b$ " is either trivially true (in case a and b are equal) or trivially false otherwise – should raise the suspicion that there are some differences between our views, in light of what has been about sentences such as "Eiffel Tower = 7", which, according to the conception of identity put forward here, is neither true nor false. Scott's main aim is to provide a formal logic (i.e. an axiomatic/deductive system) in which all complex terms can be used *on a par*, regardless of whether partial functions are involved or not. It is true that, like us, Scott takes expressions of the form " $\tau = \sigma$ " – where " τ " and " σ " are (complex) terms like " x " or " $x + 1$ " – to be inherently ambiguous, but he does this for very different reasons:

'Consider an equation like " $\tau = \sigma$ ". What should it mean? Our point of view is purely extensional, so the meaning should depend just on the 'values' of terms τ and σ and not on how they are defined (or written) syntactically. There would seem to be naturally two senses possible: (i) *both* τ and σ exist and are equal; and (ii) in *so far as* one of τ and σ exists, then so does the other and they are equal. We shall take the first as the meaning of the simple equation " $\tau = \sigma$ " because we think it is the

one more often intended. The second is important, however, and will be written “ $\tau \equiv \sigma$.”⁴¹

To achieve his aim – and to be able to enforce his ambiguity – Scott introduces (i) a distinction between ranges for *free* variables, and ranges for *bound* ones (called “quantifier scopes”), and (ii) an existence predicate “ E_s ” which he links to the notion of identity by adopting the axiom

$$(REF) \quad x =_s x \leftrightarrow E_s(x).^{42}$$

The idea (in the case of a one-sorted language) is that

- (i) the quantifier scope is included in the range of free variables, and co-extensive with the extension of “ E_s ”; and that
- (ii) the (single) \equiv -equivalence class of non- E_s (i.e. “non-existent” or “partial”) elements is to be used as the denotation for all those complex terms, like $1/0 (= x^1|_0)$, which are undefined.

In order to discuss the correlation advocated in (REF), it is necessary to establish what the *symbols* “ E_s ” and “ $=_s$ ” actually stand for, which means that we will have to look at the paradigm used by Scott and Fourman in their construction of a semantics for Scott’s logic. Before we do this, however, a brief remark concerning the logical aspects of the generalised semantic framework (see section 2): even though the interpretation of the assignments of semantic values specified by the semantic valuation rules (section 2.3) is based on the classical interpretation of the standard two-valued “valuation system” (in the sense of Dummett⁴³), according to which each sentence is “assumed to take exactly one of these truth-values, independently of our recognition of its truth-value”,⁴⁴ there is, as far as I can see, nothing which would prevent us to use the intuitive interpretation of valuation systems under which “the fundamental notion becomes that of a relativised truth-value, that of a formula’s being or not being true at a point of [a given] space S , rather than of its having one of many absolute truth-values”⁴⁵ – which lies at the heart of tense, modal and intuitionistic logics – as the basis for interpreting our assignments of semantic values. The point to remember here is that even under this sort of interpretation, the truth-valuation system would remain a proper sub-system of our assignments of semantic values: whereas the former would remain to be

⁴¹ Scott [1979], pp. 664 f.

⁴² For the sake of notational clarity, I shall add a sub-script “ S ” to “ E ” and “ $=$ ” as used by Scott.

⁴³ See Dummett [1977], pp. 164 f.

⁴⁴ Dummett [1977], p. 165.

⁴⁵ Dummett [1977], p. 167.

concerned with a formula's being true or false at a given point, the latter would have as its fundamental notion that of a relativised semantic value, that of a formula's being true, false or *non-significant* at a point of the space in question. *Semantic* valuation systems, as given by our assignments of semantic values, are always extensions of *truth* valuation systems, regardless of how the latter are interpreted.⁴⁶ It should hence not be surprising that certain conclusions which are valid in a framework based solely on truth valuations may turn out to be invalid in one based on a semantic-valuation system. Take, for example, the fact that while in the intuitionistic framework adopted by Scott⁴⁷ the non-truth of " $t = t$ " (i.e. the fact that " $t = t$ " is assigned an element of the relevant complete Heyting algebra other than its top-element) implies the non-truth of the statement that t exists, no such inference can be drawn in our framework: the fact that the sentence "The Eiffel Tower" $=_N$ the Eiffel Tower" (where " $=_N$ " designates the identity between natural numbers) is not assigned the semantic value T does not entail that the Eiffel Tower does not exist, but merely that the sentence in question is non-signification due to a "category mistake".⁴⁸

To "fix ideas", Scott and Fourman put forward the following "fundamental example" of a model for Scott's logic of partial elements:

'The set $[\dots] |R_X|$ is the collection of continuous [real-valued] maps $a: U \rightarrow \mathbf{R}$ with open domain $U \subseteq X$. We write $Ea = U = \text{dom } a$, and call this the *extent* of a . The set $Ea \in O(X)$ [the open sets of X] measures the "time" for which a "exists"; we regard a , then, as a *variable quantity* defined over X , but we have to agree that for one reason or the other a is only partially defined. *Sometimes* such partial elements can be extended to *global* elements where $a \subseteq b$ where $Eb = X$, but this is not always possible.

⁴⁶ This means, in particular, that it would be wrong to interpret the semantics specified here as a three-valued version of the Scott-Fourman semantics: even though the semantic values T and F do correspond to the top and bottom elements of Scott and Fourman's complete Heyting algebras, N cannot be identified with any value in between them, without misinterpreting what it is meant to stand for. The "truth-value gaps" indicated by an assignment of N are not of the sort given by the intermediate values of intuitionistic truth-valuation systems which reflect (the degree of) our non-recognition of the truth/falsity of sentences. They rather arise from the fact that we allow for the possibility of certain syntactically well-formed closed formulae to be *non-significant*, i.e. to be excluded from the realm of formulae which can be judged as to their truth or falsity.

⁴⁷ See Fourman & Scott [1979].

⁴⁸ This is not to say that partial elements, in the sense of Scott, could not be incorporated in our framework. Were we to base our semantic valuations on the sort of intuitionistic truth-valuations (in terms of complete Heyting algebras) adopted by Scott and Fourman, then we could well have a situation in which the phrase " $x =_k x$ " is neither assigned the top-element of the algebra, nor the value N . None the less, Scott's inference would remain to be invalid.

Given $a, b \in |R_X|$, we can measure how much they *coincide* by defining:

$$e(a, b) = \text{int}\{t \in Ea \cap Eb \mid a(t) = b(t)\}.$$

The interior operator is applied here because the properties of elements we are to be concerned with are *local* properties; thus, when $t \in e(a, b)$, the functions have to coincide in a neighbourhood of the point t . By interpreting $O(X)$ as a truth-value algebra, then $e(a, b)$ is the truth value of the statement “ $a =_{[s]} b$ ”. Note that $Ea = e(a, a)$. We call $e(.,.)$ the *equality map*.⁴⁹

Before we turn to discuss the ideas which are meant to be fixed by this example, I must point out that, in quoting Scott, I have chosen to omit certain sheaf-theoretical references for the simple reason that they are irrelevant to the meaning assigned to “ $=_s$ ” (and, implicitly, also to “ E_s ”) by this example. Having said this, it is plain that even in this revised version of the Scott-Fourman example there are still certain stipulations which are equally irrelevant in this context, the most obvious of which being the stipulation that the functions under consideration are *real-valued*. Indeed, I believe that Scott and Fourman’s conception of existence and identity can be captured without reference to topological features, which is why I propose to base our discussion on the following “function-paradigm”:

Let $|Y^X|$ be the collection of maps $a: \text{dom}(a) \rightarrow Y$ with $\text{dom}(a) \in P(X)$ – the powerset of X – where measures of “existence” and of “coincidence” are given by the following two functions:

$$E: |Y^X| \rightarrow P(X); E(a) =_{df} \text{dom}(a)$$

$$e: |Y^X| \times |Y^X| \rightarrow P(X); e(a, b) =_{df} \{t \in E(a) \cap E(b) \mid a(t) = b(t)\}.$$

The “Scott-Fourman model” based on this function-paradigm will then be given by the following additional stipulations:

- (i) $P(X)$ is to be used as truth-value algebra.
- (ii) E and e are to be used as *intuitionistically* interpreted truth-valuations for elementary E_s -expressions and elementary $=_s$ -expressions, respectively.

These two stipulations, in contrast to the ones we disregarded in formulating the function-paradigm, are crucial to the meaning acquired by “ E_s ” and “ $=_s$ ”

⁴⁹ Fourman & Scott [1979], pp. 339 f.

in the context of this paradigm. Take the case of “ E_s ”: by stipulating that E is to be used as the *intuitionistically* interpreted truth-valuation for elementary “ E_s ”-expressions, these truth-valuations are meant to be explicated in terms of the fundamental notion of being true at some point of the top-element of the truth-value algebra in question. Since, in the Scott-Fourman set-up, this top-element is nothing but the set \mathbf{X} , and since “ $E_s(a)$ ” is true at t iff $t \in E(a)$ (i.e. iff a exists at t) we can thus conclude that in the Scott-Fourman valuation

“ $E_s(a)$ ” is true iff a exists at every $t \in \mathbf{X}$.

In other words, in the Scott-Fourman model, the predicate “ E_s ” acquires the conotation (if not the meaning) of “being a *total* function”, which, given the interpretation of E as a measure of (“time of”) existence for $|\mathbf{Y}^{\mathbf{X}}|$ -individuals, is not the sort of conception of existence which one would usually adopt in the presence of such a measure, namely $ex(a)$ iff $E(a) \neq \emptyset$. Indeed, the same connotation is acquired by “ $=_s$ ”: in this case, the conclusion must be that – contrary to the standard convention concerning the use of the equality symbol – “ $=_s$ ” does *not* denote the identity relation on $|\mathbf{Y}^{\mathbf{X}}|$, for the identity of partial functions is given by their being defined at the same “times” and their having the same functional value “whenever” they are defined, i.e. by

$$Id(a,b) \text{ iff } E(a) = E(b) = e(a,b)$$

which obviously is not the same as the condition under which “ $a=_s b$ ” is true, namely iff $e(a,b) = \mathbf{X}$. Of course, if the domain of discourse is restricted to total functions, “ E_s ” and “ $=_s$ ” become *materially* equivalent with “ ex ” and “ Id ”, respectively; yet they remain, even so, different in meaning. All this is not to say that the semantics put forward by Scott and Fourman fails to be a sound and complete *formal* semantics with respect to Scott’s formal logic. My conclusion is simply that, because the meaning acquired by “ E_s ” and “ $=_s$ ” in the paradigm upon which this formal semantics is based is not that of numerical identity and existence, respectively, no conclusions about a relation between these two concepts can be drawn from the fact that (REF) is valid in all the models of the Scott-Fourman semantics.

Indeed, this emerges even more if we consider a natural generalisation of the function paradigm. Given the one striking similarity between the Scott-Fourman model (based on this function paradigm) and the sort of models we have been using to explicate our conception of identity – namely that both involve internally structured individuals – it will not be surprising that the notion of a coincidence measure, used by Scott and Fourman to fix the meaning of “ $=_s$ ”, should allow for a natural generalisation which is applicable to our sort of structured individuals, and thus provides the means of a direct com-

parison between “ $=_s$ ” and our conception of identity. The generalisation I have in mind is based on the fact that there is a natural way of representing a (partial) function $a: \text{dom}(a) \rightarrow \mathbf{Y}$ as a (partial) relation in the sense of our generalised semantical framework, namely as

$$\langle \mathbf{A} = \text{gr}(a), \text{non-}\mathbf{A} = (\text{dom}(a) \times \mathbf{Y}) \setminus \text{gr}(a) \rangle$$

– where $\text{gr}(a)$ is the *graph* of a , i.e. $\{\langle x, y \rangle \in \text{dom}(a) \times \mathbf{Y} : a(x) = y\}$. Note, in this context, that

$$i(a) =_{df} | \text{gr}(a), (\text{dom}(a) \times \mathbf{Y}) \setminus \text{gr}(a) \rangle$$

is a *one-one* function from $|\mathbf{Y}^{\mathbf{X}}|$ into the class $\mathbf{Rel}(\mathbf{X}, \mathbf{Y})$ of all *consistent* (partial) relations between elements of a class \mathbf{X} , and those of a class \mathbf{Y} (not necessarily different from \mathbf{X}) as represented in our generalised semantical framework – i.e. the class of all pairs

$$\langle \mathbf{R}, \text{non-}\mathbf{R} \rangle \text{ with } \mathbf{R}, \text{non-}\mathbf{R} \subseteq \mathbf{X} \times \mathbf{Y} \text{ and } \mathbf{R} \cap \text{non-}\mathbf{R} = \emptyset.$$

Given that the elements of $\mathbf{Rel}(\mathbf{X}, \mathbf{Y})$ will, in general, not display the sort of a-symmetry between their arguments associated with functions, the natural way of measuring them seems to be in terms of sub-sets of $\mathbf{X} \times \mathbf{Y}$ (as opposed to sub-sets of either \mathbf{X} or \mathbf{Y}). Accordingly, I suggest that we define

$$\begin{aligned} e^* : \mathbf{Rel}(\mathbf{X}, \mathbf{Y}) \times \mathbf{Rel}(\mathbf{X}, \mathbf{Y}) &\rightarrow P(\mathbf{X} \times \mathbf{Y}); \\ e^*(\langle \mathbf{R}, \text{non-}\mathbf{R} \rangle, \langle \mathbf{S}, \text{non-}\mathbf{S} \rangle) &=_{df} (\mathbf{R} \cap \mathbf{S}) \cup (\text{non-}\mathbf{R} \cap \text{non-}\mathbf{S}) \end{aligned}$$

as a measure – not of some “time-span” for which these elements co-incide – but as one of their co-incidence in the sense of overlap.⁵⁰ In other words, we are not measuring a projection of their coincidence, but the coincidence itself. The way in which e^* generalises e is that the specification of the identity between partial functions in terms of e , i.e.

$$(a) \quad a = |\mathbf{Y}^{\mathbf{X}}| b \text{ iff } e(a, a) = e(b, b), e(a, b),$$

is reflected in the way in which their relational interpretations are specified in terms of e^* , namely

$$i(a) =_{\mathbf{Rel}(\mathbf{X}, \mathbf{Y})} i(b) \text{ iff } e^*(i(a), i(a)) = e^*(i(b), i(b)) = e^*(i(a), i(b)),$$

and that *this* is *generally* true for consistent partial relation, i.e. that for all $\mathbf{R}, \mathbf{S} \in \mathbf{Rel}(\mathbf{X}, \mathbf{Y})$

⁵⁰ Note that $E^* : \mathbf{Rel}(\mathbf{X}, \mathbf{Y}) \rightarrow P(\mathbf{X} \times \mathbf{Y})$; $E^*(\langle \mathbf{R}, \text{non-}\mathbf{R} \rangle) = \mathbf{R} \cup \text{non-}\mathbf{R}$ can analogously be used to serve as a measure, not of some “time” for which $\langle \mathbf{R}, \text{non-}\mathbf{R} \rangle$ exists, but as one of the extent to which it is defined.

(A) $R =_{\mathbf{Rel}(\mathbf{X}, \mathbf{Y})} S$ iff $e^*(R, R) = e^*(S, S) = e^*(R, S)$.⁵¹

It is not difficult to see, that this “relation-paradigm” can be turned into a “Scott-Fourman model”

- (i) by using $P(\mathbf{X} \times \mathbf{Y})$ as a truth-value algebra, and
- (ii) by interpreting $e^*(R, S)$ as intuitionistic truth-valuation for “ $R =_S S$ ”;

and that the suggested conception of a coincidence measure for consistent partial relations applies directly to the structured individuals of a consistent domain, say \mathbf{D}_k^δ – generated by some identity predicate Id_k on the basis of some reference manifold \mathbf{C} – in virtue of the fact that $\mathbf{D}_k^\delta \subseteq \mathbf{Rel}(\mathbf{C}^\delta[Id_k], \mathbf{C}^\delta[Id_k])$. Let $\mathcal{M} \in \mathbf{Id}_{\mathbf{C}^\delta}$ (which means that “ Id_k ” is interpreted as the identity predicate of \mathbf{D}_k^δ), and \mathcal{M}^* the “Scott-Fourman model” based on $\mathbf{Rel}(\mathbf{C}^\delta[Id_k], \mathbf{C}^\delta[Id_k])$ and e^* . Given this, we find that for any $\sigma \in \mathcal{M}$

“ $Id_k(x, y)$ ” is *true* in \mathcal{M} iff $\sigma(x) =_{\mathbf{Rel}(\mathbf{X}, \mathbf{Y})} \sigma(y)$.

Indeed, if we focus on “classical” (i.e. consistent and complete) domains, it becomes clear that

“ $Id_k(x, y)$ ” is *false* in \mathcal{M} iff $\sigma(x) \neq_{\mathbf{Rel}(\mathbf{X}, \mathbf{Y})} \sigma(y)$,

which shows that, in \mathcal{M} , “ Id_k ” *does* denote the relevant identity relation. In the case of \mathcal{M}^* , however, we find – since for any $\sigma \in \mathcal{M}^*$

“ $x = y$ ” is *true* in \mathcal{M}^* iff $e^*(\sigma(x), \sigma(y)) = \mathbf{C}^\delta[Id_k] \times \mathbf{C}^\delta[Id_k]$

– that for any $\sigma(x), \sigma(y) \in \mathbf{D}_k^\delta$

“ $x =_S y$ ” is *true* in \mathcal{M}^* iff $\text{card}(\mathbf{D}_k^\delta) = 1$,

which, given that we can easily define domains with more than one element, can only mean that, as before, $=_S$ is *not* the identity relation of the domain of discourse it applies to.

As a final remark in this section let me point out that there is an interpretation of the language used by Scott within our semantic framework which does not involve non-existent objects, and which – in contrast to the semantics put forward by Scott and Fourman – actually does reflect Scott’s initial motivation

⁵¹ Note that if (a) is accepted to be a special case of (A), then (a) must be taken as the characterisation of the identity between partial functions, because (A) is not true by convention, but follows from the fact that $=_{\mathbf{Rel}(\mathbf{X}, \mathbf{Y})}$ is nothing but the identity on $\mathbf{X} \times \mathbf{Y}$.

that we should be allowed to “feel free to use complex expressions (terms) without demanding that they always denote”.⁵² The basic idea is simply

- (i) to interpret Scott’s existence predicate not as a *monadic* predicate about individuals, but as a *dyadic* predicate relating individuals and (partial) functions: instead of interpreting “ $E(a(t))$ ” as “ $a(t)$ exists”, it is to be read as “ a exists at t ”,⁵³ and
- (ii) to evaluate all other elementary predicate expressions as being *non-significant* if they contain a complex term “ $a(t)$ ” which is non-denoting, meaning that a does not exist at t .⁵⁴

The “logic of partial functions”, i.e. the logic of the language resulting from this interpretation, I believe captures precisely the sort of reasoning which Scott had in mind when he developed his logic. Indeed, there are reasons to believe that it actually *is* (possibly a slightly stronger, but none the less non-classical version of) his “logic of partial elements”.⁵⁵

5.2. The Construction of Real Numbers

Scott argues, that the generality which his formal system has over the ordinary (first-order intuitionistic) logic is not only interesting but indeed necessary “because there is simply no way of avoiding the passage from a structure to a *substructure*. Formally this passage can be expressed by the relativisation of the quantifiers to the predicate defining the substructure. [...] If we wish to reason about the substructure, the more general logic is seen as entirely appropriate.”⁵⁶

As a first example of a relativisation of quantifiers, Scott puts forward the definition of the real numbers in terms of Cauchy sequences:

⁵² Scott [1979], p. 662. In the Scott-Fourman semantics, all well-formed (closed) terms denote, although some reference objects may be non-existing individuals.

⁵³ This means, of course, that “ $E(t)$ ” must be read as a short form of “ $E[id(t)]$ ”, where id is the identity function $id(x) = x$ on the domain of “ t ”. Note, incidentally, the resulting dependence of existence on identity, if the latter is conceived of in the way suggested in this essay.

⁵⁴ ad (i) “ $E[a(t)]$ ” is *true (false)* in M iff $E(a)$ is part of the domain of M which includes t and $t \in E(a)$ [$t \notin E(a)$] – and *non-significant* otherwise.

ad (ii) “ $P[a(t)]$ ” is *true (false)* in a model M iff $t \in E(a)$ and $a(t) \in \mathbf{P}^M$ [$a(t) \in \mathbf{non-P}^M$] – and *non-significant* otherwise.

⁵⁵ The models – as far as one can judge from the way we have sketched them – are sound with respect to Scott’s logic, which means that the logic of the language in question will be at least as strong as his one. Its non-classical nature, on the other hand, follows from the fact that if “ $a(t)$ ” does not denote in a model M , then “ $P(a(t))$ ” will be *non-significant* in M , and hence $M \not\models P[a(t)] \vee \neg P[a(t)]$ for any predicate “ P ” – the assumption being that $(N \vee \neg N) \leftrightarrow N$.

⁵⁶ Scott [1979], p. 678.

- [A] “We are assuming as known the rationals, \mathbf{Q} , with their usual structure. Let \mathbf{Q}^∞ be the space of (simple) infinite sequences, $\langle x_n \rangle_{n=1}^\infty$, of rationals; that is, we assume enough (intuitionistic) set theory to be able to do a completion by sequences. We regard \mathbf{Q}^∞ as a perfectly nice set, where two sequences are equal if and only if they are termwise equal. [...] In the well-worn manner we are going to single out a subset of \mathbf{Q}^∞ , call it \mathbf{S} , and the Cauchy reals, \mathbf{R}^c , will be a quotient of \mathbf{S} by an equivalence relation. [...] In fact \mathbf{S} is easily defined in terms of the equivalence relation: for $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ define

$$x \approx y \text{ iff } |x_n - y_m| \leq 1/n + 1/m \text{ for } n, m = 1, 2, 3, \dots$$

This is not an equivalence relation on all of \mathbf{Q}^∞ but only on the subset defined by

$$x \in \mathbf{S} \text{ iff } x \approx x.$$

Thus a Cauchy real number (generator) is given by a sequence with modulus of convergence $1/n$. Without much trouble we see that the relation is symmetric and transitive. [...] But to prove that x is a generator, we have to prove $x \in \mathbf{S}$; the real number only *exists* when it is given by a *convergent* sequence. *Existence* for reals means $x \in \mathbf{S}$. [...]

In this example, because an equivalence relation is involved, it is perhaps not quite so clear how quantifiers are relativized. The point is, of course, that starting out with ‘ordinary’ logic on \mathbf{Q}^∞ , to get the theory of \mathbf{R}^c we need to replace $=$ by \approx . This introduces partial elements, because $x \approx x$ does not hold throughout \mathbf{Q}^∞ . Even if we relativize to $\mathbf{S} \subseteq \mathbf{Q}^\infty$ it does not at once obviate the question, since we can define operations under which \mathbf{S} may not be closed. [...]

- [B] It is more elementary to use elements in such cases instead of classes [...], but the language of classes [...] makes the act of relativization particularly simple. Let $P\mathbf{Q}^\infty$ be the powerset of \mathbf{Q}^∞ , a domain on which we can use ordinary logic. We define $\mathbf{R}^c \subseteq P\mathbf{Q}^\infty$ as the class of equivalence classes:

$$\mathbf{R}^c = \{X \subseteq \mathbf{Q}^\infty \mid \exists x \in \mathbf{S} \forall y [y \in X \leftrightarrow x \approx y]\}$$

[...] we regard all quantifiers such as ‘ $\exists X$ ’ as *relativised* to \mathbf{R}^c (that is, replaced by ‘ $\exists X \in \mathbf{R}^c$ ’).⁵⁷

⁵⁷ Scott [1979], pp. 678f.

It is easy to see that Scott's initial specification of Cauchy-reals (given under [A]) lends itself to be interpreted as the generation of a domain of structured individuals, as represented in the models of our structurally enriched semantics. All we need to do to is:

- (i) to take Scott's set \mathbf{S} as a reference manifold (i.e. as a class of "generators") to which a certain identity predicate, say Id_r , is applied, and
- (ii) to interpret \approx as representing the basic Id_r -identifying acts (see section 4.1) on \mathbf{S} ; i.e. define the elementary identifying function δ for Id_r on \mathbf{S} as follows:

$$\mathbf{Id}_r^\delta =_{df} \{ \langle x, y \rangle \in \mathbf{S} \times \mathbf{S} : x \approx y \} \text{ and } \mathbf{non-Id}_r^\delta =_{df} \mathbf{S} \times \mathbf{S} \setminus \mathbf{Id}_r^\delta.$$

The structured reference manifold \mathbf{S}^δ then gives rise to a domain \mathbf{D}_r^δ corresponding to the (set of) Cauchy reals in the sense of being isomorphic (as identity structure) to \mathbf{R}^c , i.e. of there being a set-theoretic mapping $\varphi : \mathbf{D}_r^\delta \rightarrow \mathbf{R}^c$ which is one-one and onto, such that for any structurally enriched model \mathbf{M} (based on \mathbf{D}_r^δ) in which " Id_r " is indeed interpreted as an identity predicate – i.e. for any $\mathbf{M} \in \mathbf{Id}_{\mathbf{S}^\delta}$ – we have that (for all $\sigma \in \mathbf{M}$)

$$v(\langle Id_r(x, y), \mathbf{M}, \sigma \rangle, T) \text{ iff } \varphi(\sigma(x)) =_\epsilon \varphi(\sigma(y)).^{58}$$

In order to construct the model, say \mathbf{R}^c , corresponding to the Cauchy reals in our enriched framework, all we would now need to do is to specify the operations of addition, subtraction, multiplication, and division for the individuals of the domain \mathbf{D}_r^δ in analogy to the way in which they are introduced in the classical definition.⁵⁹ It should be noted, that – even though the elements of $\text{dom}_{\mathbf{R}^c}^\delta(Id_r)$ (i.e. of \mathbf{D}_k^δ) are set-theoretical constructs based on $\mathbf{S} \subset \mathbf{Q}^\infty$ – the identity relation (given by $\langle \mathbf{Id}_k^{\mathbf{R}^c}, \mathbf{non-Id}_k^{\mathbf{R}^c} \rangle$) of the *object-language* (say $L_{\mathbf{R}^c}$) of \mathbf{R}^c is, as far as $L_{\mathbf{R}^c}$ is concerned, *not* reducible to the identity relation between the elements of \mathbf{Q}^∞ involved in the relevant constructions. The "generators", i.e. the members of $\mathbf{S} \subset \mathbf{Q}^\infty$, are simply not accessible to $L_{\mathbf{R}^c}$. Obviously, this is not meant to imply that one cannot specify denotations for $L_{\mathbf{R}^c}$ -predicate and function symbols by referring to the internal components of the elements of \mathbf{D}_k^δ and by using the properties, relations and operations defined for them as members of \mathbf{Q}^∞ , but only that such specifications are, as far as $L_{\mathbf{R}^c}$ is concerned, meta-linguistic in character.

⁵⁸ Note that since Id_r is classical in all $\mathbf{M} \in \mathbf{Id}_{\mathbf{S}^\delta}$, we also have that (for all $\sigma \in \mathbf{M}$)

$$v(\langle Id_r(x, y), \mathbf{M}, \sigma \rangle, F) \text{ iff } \delta(x) \neq_\epsilon \sigma(y)$$

which implies that \mathbf{D}_k^δ and \mathbf{R}^c are indeed equivalent set-theoretical sense.

⁵⁹ For example: $\langle \mathbf{Id}_k(\langle x_n \rangle), \mathbf{non-Id}_k(\langle x_n \rangle) +_{\mathbf{R}^c} \langle \mathbf{Id}_k(y_n) \rangle, \mathbf{non-Id}_k(\langle y_n \rangle) \rangle =_{df} \langle \mathbf{Id}_k(\langle x_{2n} +_Q y_{2n} \rangle), \mathbf{non-Id}_k(\langle x_{2n} +_Q y_{2n} \rangle) \rangle$.

Returning to Scott's example, it seems to be reasonable to see the passage he refers to as "starting out with 'ordinary' logic on \mathbf{Q}^∞ , to get the theory of \mathbf{R}^c " as embedded in the transition from a particular first-order language interpreted in \mathbf{Q}^∞ , to the (first-order) "counterpart language" interpreted in the Cauchy reals. The most basic case of such a transition – within our semantic framework – is, of course, that from the "language of \mathbf{Q}^∞ -identity" $L_{Q^\infty}^-$ (i.e. the first-order language with " Id_q " as only predicate symbol) interpreted in the structurally enriched model \mathbf{Q}^∞ corresponding to \mathbf{Q}^∞ , to the "language of the identity of Cauchy reals" $L_{R^c}^-$ (with " Id_r " as only predicate symbol) interpreted in \mathbf{R}^c ; and it is quite sufficient for the purposes of the present discussion to focus on this particular transition. The best, if not the only way of comparing these two languages is by "embedding" them in a suitably interpreted joint-language, that is a (first-order) language L^* (with a vocabulary which includes the vocabularies of $L_{Q^\infty}^-$ and $L_{R^c}^-$), interpreted in a structurally enriched model $4M^*$ which retains the interpretations of \mathbf{Q}^∞ and \mathbf{R}^c . What can we say about M^* ? The domains of \mathbf{Q}^∞ and \mathbf{R}^c will somehow have to be embedded in the domain(s) of M^* . The way in which this is to be done, however, depends on the question whether the Cauchy reals (as represented by $dom_{R^c}^\delta(Id_r)$) are of the same *category* (in the sense introduced in this essay) as the elements of \mathbf{Q}^∞ . Given that the identity relation on \mathbf{Q}^∞ is defined by

$$a =_q b \text{ iff}_{df} \text{ for every natural number } n, \\ \text{the } n\text{-th component of } a =_Q \text{ the } n\text{-th component of } b;$$

it is not difficult to see that this is not the case: the sentence " $r =_q s$ " is non-significant if " r " or " s " refers to a Cauchy-real (regardless, incidentally, of whether one takes Cauchy reals to be individuals generated by identifications, or \approx -equivalence classes $\in PS$), for they simply lack the sequential nature required to apply the phrase " n -th component of" meaningfully.

The domains of M^* representing those of \mathbf{Q}^∞ and \mathbf{R}^c will thus have to be essentially disjoint domains, a fact which requires us to treat the joint-language L^* as a two-sorted language – one sort associated with Id_q and the variables of $L_{Q^\infty}^-$, the other with Id_r and the variables of $L_{R^c}^-$ – if we wish to avoid an unnecessary proliferation of non-significance. The transition from the language of \mathbf{Q}^∞ -identity to the language of the identity of our Cauchy reals does hence *not* involve a passage from a structure to a sub-structure (formally expressed by a Scott-type relativisation of quantifiers), but rather a passage from a (sub-) structure of one category to another one of an *incompatible* category, formally expressed by a replacement not only of " Id_q " by " Id_r ", also by a re-

placement of all $L_{Q^\infty}^\infty$ -variables by (suitably chosen) $L_{R^c}^\infty$ -variables.⁶⁰ This, I believe, sheds some light on the real reason why “it is perhaps not quite so clear how quantifiers are relativised” in the transition from ‘ordinary’ logic on Q^∞ to the theory of R^c , namely simply that no such relativisation takes place. This, of course, is not meant to imply that no relativisation takes place in the example put forward by Scott under [B]; the point is rather that we are in this case dealing with a transition different from the one occurring in [A], and that the link which Scott employs in explicating the putative relativisation in [A] by means of the unquestionable relativisation in [B] simply does not exist. There is no doubt that there are elements $x \in Q^\infty$ for which “ $x \approx x$ ” is not true; yet, contrary to Scott’s claim, this does *not* introduce these elements (i.e. the elements of $Q^\infty \setminus S$) as partial elements, even if we replace the Q^∞ -identity predicate by “ \approx ”.⁶¹ And even if it did, this would still not warrant Scott’s implicit assumption that in the language of classes, the elements of $PQ^\infty \setminus R^c$ correspond to the elements of $Q^\infty \setminus S$ in the way in which those of R^c correspond to those of S : the elements of $Q^\infty \setminus S$ are not only *not* generators of Cauchy-reals as conceived by Scott (i.e. of a certain *kind* of subsets of $Q^\infty \setminus S$), as far as \approx is concerned, they simply *do not generate any sets at all*. This is not to say that there might not be situations in which it would indeed be necessary to use Scott’s logic, but merely that the generation of Cauchy-reals fails to exemplify such a situation.

More importantly, however, our interpretation of the generation of Cauchy-reals will have illustrated the fact that *incompatible* sorts are not confined to higher-order languages – say, as formalised by Scott,⁶² where they arise through his usage of power sets and product sets as primitives⁶³ – but they can occur in *any* language (with identities). This means, in particular, that if, say for logical purposes, we were interested in augmenting our syntax with rules that eliminate the possibility of forming non-significant formulae,

⁶⁰ In fact, this picture of what the said transformation amounts to will also emerge from this analysis if the traditional conception of Cauchy-reals as equivalence classes is used, and the joint-language is taken to be the relevant language of construction, i.e. the inherently two sorted language of classes.

⁶¹ The fact that “ $x \approx x$ ” is not true would only imply the partial character of x if \approx were the relation of strict identity (in Scott’s sense) on the range of the (free) variable “ x ”, and this could only be the case if it were co-extensive on what Scott seems to identify as the scope of the bound variable “ x ”, namely S , with the ordinary Q^∞ -equality relation. But this, it clearly is not, since there are Q^∞ -unequal $a, b \in S$ with $a \approx b$.

⁶² See Scott [1979], pp. 685 ff.

⁶³ Even though Scott stresses that “as sorts are *symbols*, equality between sorts means notational equality and not identity under semantical interpretation” [1979, p. 687], it is clear that, say, in the case of a sort A and its power sort $P(A)$, the notational difference reflects an incompatibility in the sense that the respective semantical interpretations must (apart from possibly the empty set) be disjoint.

we would in general not only have to adopt a sorted language, but a “category-valued” one. The augmented syntax would not only depend on a sortal classification of the terms and formulae (as represented by Scott’s #-function), but also on a *category classification* of its sorts, representable by a “category valuation function” $c: \mathbf{Sort} \rightarrow \{1, 2, 3, \dots\}$. It is worth pointing out in this context that a category-valued (many-sorted) *first-order* language (as interpreted in our structurally enriched semantics) can generally *not* be reduced to a one-sorted language in the way in which this can be achieved in the traditional framework:⁶⁴ if the language involves sorts, say A and B, of different categories ($c(A) \neq c(B)$) then A and B will always be interpreted as essentially disjoint sets, which, as such, cannot be construed as sub-sets of a single domain.

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⁶⁴ See, for example, Barwise [1977], p. 42.